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10 Dr. Alfred I. Kaufman

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1. Enclosure (1) is forwarded as a matter of possible interest.
2. Recently, the Center for Naval Analyses has initiated a continuing investigation into our present capability of finding submarines by acoustical means, with the expectation that such an effort will provide a more accurate modeling of the search process and, perhaps, will suggest to the Navy better ways of performing its antisubmarine warfare task. In this first publication of the Acoustic Detection Project, we report on our current understanding of the acoustic search problem and indicate the direction we intend pursuing next.
3. Your comments are solicited regarding the ideas contained within this memorandum.

DAVID A. PERIN
Director
Naval Warfare Analysis Group

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ABSTRACT

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We present a critical review of the fundamental ideas that underlie the conventional theory of acoustic detection. This review is constructed around the suggestion that acoustic detection apparatus be employed for collecting information about a complete set of tactically relevant parameters rather than for testing hypothesis. This idea is illustrated by applying the Woodward & Davies concept of sample-path information to the problem of detection of signals in white noise. Finally, we argue that within such an information-theoretic framework, information generally disregarded by conventional procedures can be extracted from the received voltage history. To obtain this additional amount of information, it is required that we possess a far more detailed description of underwater acoustics than what a conventional theory would demand. In return, however, passive range measurements based on the received voltage history alone become conceptually feasible.
↑

SECTION I INTRODUCTION

THE SEARCH PROBLEM

Recognizing the important role that searching for submarines plays in a major naval encounter, the Center for Naval Analyses has initiated a continuing investigation into our present capability of finding submarines by acoustical means, with the expectation that such an effort will provide a more accurate modeling of the search process and, perhaps, will suggest to the Navy better ways of performing its antisubmarine warfare task. In this first publication of the Acoustic Detection Project, we report on our current understanding of the acoustic search problem and indicate the direction we intend pursuing next.

The observation fundamental to our discussion is that ocean phenomena, to the extent they affect sound-wave propagation through the ocean environment, are the determining factor in the effectiveness with which naval forces search for an acoustic target. In fact, far from being an unqualified study of ocean physics, our investigation is explicitly aimed at understanding the manner in which our capability to conduct a successful search is influenced by the physical processes occurring in the ocean. Therefore, a meaningful positing of the problem of ocean modeling requires that we first embed this problem into a larger mathematical framework dealing with the transformation of measured acoustic pressure changes into tactically useful information. Without a processing scheme clearly defined, there exists no meaningful way of

determining the type of description of ocean phenomenology that is relevant to our stated purpose.

In conventional processing, the acoustic pressure changes or equivalently, the voltage history measured at the output of the hydrophone system are considered as sample values used to statistically determine whether or not a given signal is contained within the received acoustic pressure field. This puts conventional processing squarely into the realm of statistical hypothesis testing and, hence, requires that the processor perform a thresholding operation upon a suitably chosen function of the voltage history (reference 1). If target bearing and frequency spectrum are also desired, the processor is allowed, before the thresholding operation, to perform the appropriate Fourier transformation upon the received voltage history. Other parameters of tactical relevance, notably the range to the target, are distinctly harder to ascertain within the conventional framework and, in fact, modern processors are not capable of performing a reliable passive ranging operation.

As we shall show in some detail later, this processing philosophy demands that we model the ocean by providing a description of the statistical properties of the signal-to-noise ratio:

$$\rho^2 = \frac{2E}{N_0} ,$$

where E represents the signal energy content over the sample length considered and N is the spectrum density of the noise power.

We want to argue that a more appropriate processing framework is obtained if we identify processing with the gathering of information about tactically relevant parameters rather than with statistical hypothesis testing. Clearly, to render this idea mathematically meaningful, a precise definition of what is mean by information must be provided and we shall do so presently. It will become apparent from the definition given to the concept of information, that the amount of information one has about a given parameter is a monotonic function of the statistic a conventional processor would evaluate. Hence, in conventional processing, information that comes below the chosen threshold is entirely disregarded; conventional gear appears to be operating with less information than is actually available in the random voltage history. Furthermore, since within this framework one would evaluate information about all tactical parameters alike, passive ranging is no longer conceptually different from any other part of the detection process.

If we allow this information-theoretic framework of signal processing to determine what features of the ocean environment should be modeled, we find that we now need the probability density function of the sample voltage measurement conditioned upon the value taken by the tactical parameter of interest; this is a fare more detailed description of the ocean than that required by the conventional philosophy, which is as it should be for we expect this processing procedure to provide us with more information about relevant parameters than the conventional processors do.

Due to the complexity of the problem at hand, we only claim to have argued that the information-theoretic approach is reasonable. We hope to show later that it is also useful in that it provides the searchers with an increased capability to find the target submarine.

THE CONCEPT OF INFORMATION

In what follows, we shall take the position that the only element of communication between the searcher and the submarine he searches for is the random voltage history that obtains at the outlet of his hydrophone system. Depending on how he chooses to look at it, this voltage history will provide him with information concerning various parameters of tactical usefulness to his search effort. But before we discuss the details of how this is done, it behooves us to specify, and subsequently quantify, the concept of information we intend to employ here. Following the 1952 work of Woodward & Davies (reference 2), let ξ represent the tactical parameter of interest, $V_0^t \equiv \{v(\tau), \tau \in [0, t]\}$ the voltage history that the searcher has measured over the $[0, t]$ time interval, and $U(\hat{\xi}|V_0^t)$ the searcher's state of knowledge at time t concerning a particular value $\hat{\xi}$ given the specific voltage sample path V_0^t . The conventional, and perhaps the most convenient representation for one's state of knowledge is obtained if one is willing to consider ξ a random variable and $U(\hat{\xi}|V_0^t)$ the corresponding conditional probability density given the measurement V_0^t .

When a measurement has been performed, values of the tactical parameter that would imply a high probability for having measured V_0^t will appear to the observer more believable than other possible values. Due to distortion by the

random medium through which the message has propagated, this reevaluation of beliefs will not lead to an a posteriori state of knowledge entirely concentrated on a single value of ξ . Rather, the searcher's state of knowledge concerning the value of some tactical parameter after having read the voltage history V_0^t will merely differ from his a priori state of knowledge $U(\xi|V_0^0)$. This, according to Woodward & Davies, is the essence of the acoustic detection process.

It seems intuitively reasonable that the amount of information contained within a measurement V_0^t should be related to the change that measurement produced in the observer's state of knowledge. To quantify this notion, Woodward & Davies require the following statements to be true:

- If 2 successive voltage readings are employed to learn about the value of some tactical parameter, and the observer regards his a posteriori state of knowledge after the first reading as the a priori one before the second, the total gain in information concerning the value of the parameter is equal to the sum of the gains from each reading.
- If a voltage reading is received and used to learn about the value of 2 independent parameters, the total gain in information concerning them is the sum of the gains when each parameter is considered separately.

From these 2 axioms, Woodward & Davies develop the whole mathematical theory. Their argument proceeds as follows. If upon receiving a reading V_0^t the observer is certain that the value of the tactical parameter is $\hat{\xi}$, the amount of information contained in that reading depends only on the observer's initial state of knowledge $U(\hat{\xi}|V_0^0)$. The first axiom then implies that the

amount of information contained in a voltage reading which leaves the observer with some uncertainty as to the actual value of ξ is a function of the a priori and a posteriori states of knowledge about that value alone and may be written in the form $J[U(\hat{\xi}|V_0^0), U(\hat{\xi}|V_0^t)]$. With this notation, the first axiom states:

$$J[U(\hat{\xi}|V_0^0), U(\hat{\xi}|V_0^{t_1})] + J[U(\hat{\xi}|V_0^{t_1}), U(\hat{\xi}|V_{t_1}^{t_2})] \\ \equiv J[U(\hat{\xi}|V_0^0), U(\hat{\xi}|V_0^{t_2})] .$$

Woodward & Davies show that to satisfy this identity, J must be of the general form

$$J(\alpha, \beta) = j(\alpha) - j(\beta) .$$

In order to determine the functional form of j , the second axiom is used. Here the 2 independent parameters may be denoted by ξ and η , and the corresponding received voltage indication again by V_0^t . If the state of knowledge concerning the joint entity of 2 independent tactical parameters is taken to be the product of the states of knowledge concerning them separately, the second axiom gives

$$\begin{aligned}
& j[U(\hat{\xi}|V_0^0) U(\hat{\eta}|V_0^0)] - j[U(\hat{\xi}|V_0^t) U(\hat{\eta}|V_0^t)] \\
& \equiv j[(\hat{\xi}|V_0^0)] - j[U(\hat{\xi}|V_0^t)] + j[U(\hat{\eta}|V_0^0)] - j[U(\hat{\eta}|V_0^t)] .
\end{aligned}$$

From this identity, $j(U)$ must be of the form

$$j(U) = A \log U + B$$

where A and B are constants which may be chosen arbitrarily. Thus,

$$j[U(\hat{\xi}|V_0^{t_1}), U(\hat{\xi}|V_0^{t_2})] = \log \frac{U(\hat{\xi}|V_0^{t_2})}{U(\hat{\xi}|V_0^{t_1})} ,$$

where A was so chosen as to make an increase in the observer's state of knowledge represent positive information. This result of Woodward & Davies is the expression we shall use in this paper to represent the quantity of information gained from $V_{t_1}^{t_2}$ concerning the value of the tactical parameter ξ . As might be expected, it is equal to zero if, and only if $U(\hat{\xi}|V_0^{t_2})$ and $U(\hat{\xi}|V_0^{t_1})$ are identical, i.e., when the communication $V_{t_1}^{t_2}$ leaves the observer's state of knowledge about $\hat{\xi}$ completely unchanged.

SUMMARY

In section II, we shall briefly discuss those factors in the submarine search problem that are responsible for the random nature of the voltage

received, and shall, hence, mention the major aspects of underwater acoustic propagation. In section III we present a number of various ways in which the received voltage can be manipulated and we evaluate the amount of information that can thus be obtained. In section IV, we give a general description of how conventional modeling employs this information to evaluate Navy's tactical performance. We conclude this paper with a critical evaluation of conventional models and delineate some research topics we intend addressing within the Acoustic Detection Project.

SECTION II

THE RANDOMNESS OF THE OCEAN ENVIRONMENT

As already outlined in the preceding section, the main input to the acoustic detection process is the random voltage measured at the output of the observer's hydrophone system. We have also argued that the essence of acoustic detection is the change that this random voltage continuously induces in the observer's state of knowledge regarding any tactically relevant parameter. That this state of knowledge should be imperfect is a consequence of the random nature of the received voltage and the finite nature of the observation interval. It is therefore essential that we attempt to understand the physical reasons for this randomness.

Possibly the most important contributor to the random nature of the acoustic voltage is the ocean medium through which the submarine signal must propagate to arrive at the observer's hydrophone system. Indeed, the physical properties of the ocean at every point within it undergo irregular fluctuations. Similarly, the physical properties of the ocean at different spatial points within it, but at the same instant in time differ from one another in a random fashion. In particular, since the acoustic index of refraction of the ocean is a function of such physical properties as temperature and salinity, one can take the view point that the refractive index is a random field. There are many different mechanisms that have been proposed for modeling the random properties of the acoustic refraction index of the ocean (references 3, 4, and 5).

It appears unlikely, however, that any one such model will eventually account for all of the observed features of acoustic fluctuations in the ocean. Rather, we must rely in practice upon a mixture of limited models, in which one mechanism might dominate over the others under specified conditions, but none suffices to account for underwater acoustic behavior under all conditions.

The conventionally accepted procedure for translating the statistical properties of the random ocean into the implied statistical properties of an acoustic pressure field supported by such a medium is to use the wave equation with suitably designed boundary conditions. Quite often, these boundary conditions themselves are random which only aggravates the already formidable task of finding solutions to the wave equation. Due to the mathematical complexity of this approach, a number of approximate solutions to the wave equation under simplifying assumptions about the acoustic process have been developed and used over the years (references 6, 7, and 8).

It is not our intention to review, at this time, progress in the field of acoustic propagation in random media in any detail; the complexity of the problem, as well as the amount of professional literature that has been written to date, precludes such an attempt. We only wish to observe that whatever the accepted theory of underwater sound propagation, it should provide the probability distribution of the acoustic voltage history measured at the hydrophone output in terms of all those parameters that are deemed necessary to specify such distribution entirely. Since a parameter that does not in any way affect

this probability cannot be estimated by voltage measurements, the set of parameters necessary to specify the voltage probability distribution will be referred to as a complete set of tactically relevant parameters. We shall assume in what follows that such a probability distribution is somehow made available to us, and proceed to evaluate the amount of information regarding tactically relevant parameters that is contained within a sample voltage measurement.

SECTION III

THE INFORMATION-THEORETIC APPROACH TO DETECTION

THE EXISTENCE PARAMETER

The most straightforward, although not necessarily the most complete, application of Woodward & Davies' concept of sample-path information to problems of signal detection appears in a 1952 paper by Davies (reference 9). In that work, the important tactical parameter is taken to be a 2-valued existence parameter, ϵ , designed to represent either the presence, $\epsilon = \text{YES}$, or non-presence, $\epsilon = \text{NO}$, of a specified acoustic signal in the received voltage history. The observer's state of knowledge concerning this parameter is represented by the conditional probability $p(\epsilon|V_0^t)$ that the signal be present within the received voltage history given the sample path V_0^t . To evaluate the time dependence of this probability along a voltage sample path, the theory relates it via Bayes' rule to the conditional probability that the sample path under consideration would be obtained given a particular value of the existence parameter,

$$p(\epsilon|V_0^t) = \frac{p(\epsilon|V_0^0) p(V_0^t|\epsilon)}{p(\text{YES}|V_0^0) p(V_0^t|\text{YES}) + p(\text{NO}|V_0^0) p(V_0^t|\text{NO})} \quad (1)$$

where the probability $p(\epsilon|V_0^0)$ represents the observer's a priori state of knowledge regarding the value of the existence parameter ϵ . Therefore, the amount of information regarding the presence of a target signal within the

voltage history V_0^t is given, in accordance with the definition provided in the previous section, by

$$J[p(\epsilon|V_0^0), p(\epsilon|V_0^t)] = \log \frac{p(\epsilon|V_0^t)}{p(\epsilon|V_0^0)},$$

and using equation (1),

$$J[p(\epsilon|V_0^0), p(\epsilon|V_0^t)] = \log \frac{p(V_0^t|\epsilon)}{p(\text{YES}|V_0^0)p(V_0^t|\text{YES}) + p(\text{NO}|V_0^0)p(V_0^t|\text{NO})}. \quad (2)$$

Equation (1) mathematically specifies the ideal receiver for estimating the presence of a signal within the wave form V_0^t . Within an information theoretic approach to detection, all that can be reasonably demanded of a receiver is that it shall enable an observer to determine probabilities that each possible value of the parameter of interest is the true one. If the receiver actually computes these probabilities, no further problem of interpreting the content of the wave form regarding the desired tactical parameter remains.

THE CASE OF A DETERMINISTIC SIGNAL

In order to apply the foregoing theory to a practical problem, the illustrative value of which might be quite welcome at this point, it is necessary to consider how the a posteriori state of knowledge $p(\epsilon|V_0^t)$ is constructed

from the received communication V_0^t . We shall presume that the a priori state of knowledge $p(\epsilon|V_0^0)$ is given and the a posteriori distribution may then be obtained directly from equation (1) once $p(V_0^t|\epsilon)$ has been evaluated.

The conditional probability $p(V_0^t|\text{YES})$ represents the unpredictable nature of the received wave form V_0^t when the target signal is known to be present and, hence, describes the effect various random disturbances present in the reception process have upon the shape of the signal wave form. In general, these random disturbances can be classified either as distortions in the signal wave form that are mainly due to randomness in the propagation medium, or as interferences from sound sources other than the target such as ambient sound producers, thermal noise in the receiving electronic apparatus and altogether everything else that might contribute to the conditional probability $p(V_0^t|\text{NO})$.

In the following example, signal wave form distortions will not be considered. Rather, we shall assume that the signal wave form S_0^t is exactly known and that the voltage history received in the presence of a target differs from S_0^t only in that a random wave form, generally referred to as noise, has been added to it. The large variety of contributory phenomena make the noise wave form and its statistical properties very difficult to describe mathematically. To make our example manageable, we shall represent the noise by a stationary, band-limited, white, Gaussian random process. Under these conditions, the evaluation of $p(V_0^t|\epsilon)$ is fairly straightforward and will now be described.

A simple way of formulating the statistical properties of the noise process is by means of wave form sampling process. Sampling analysis rests on a well-known mathematical theorem (reference 10) that if a function of time $F(t)$ contains no frequencies greater than W , then over any finite observation time T

$$F(t) = \sum_{j=0}^{2WT-1} F\left(\frac{j}{2W}\right) \frac{\sin \pi (2WT - j)}{2WT \sin \pi \left(\frac{t}{T} - \frac{j}{2WT}\right)} .$$

The importance of this identity is that it enables a continuous function of time to be specified uniquely in terms of sample values at intervals $\frac{1}{2W}$, where W is an arbitrary frequency greater than any which occurs in the frequency spectrum of $F(t)$.

The normality of the assumed noise process implies that at the output of the filter, each sampling value, $n\left(\frac{j}{2W}\right)$, or more briefly n_j , has by definition the Gaussian probability density

$$p(n_j) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{1}{2}} e^{-n_j^2/2\sigma^2}$$

where σ^2 is the mean square value of $n(t)$. It is not difficult to show that these noise samples are statistically independent. Indeed, under the stationarity assumption, the correlation function of the noise process

$$\phi(t, t+\tau) \equiv \langle n(t) n(t+\tau) \rangle ,$$

where brackets indicate that the average over an ensemble is to be taken of the quantity within, becomes a function of τ alone (reference 11). Its Fourier transform with respect to τ ,

$$\phi(\tau) = \int_{-\infty}^{\infty} d\nu e^{-i2\pi\nu\tau} \phi(\nu) \quad (3)$$

represents the noise power spectrum (reference 11) and, hence, with our whiteness assumption, can be written as

$$\phi(\nu) = \begin{cases} \frac{1}{2}N_0 & |\nu| < W \\ 0 & \text{otherwise} \end{cases}.$$

Performing the integration indicated in equation (3),

$$\phi(\tau) = N_0 W \frac{\sin 2\pi W\tau}{2\pi W\tau},$$

and noise samples taken $\frac{1}{2W}$ apart are indeed statistically uncorrelated. Therefore, the joint probability density of a set of samples is the product of each separate distribution. Since the samples determine the wave form, this product also gives the probability density for the wave form itself. With

$$\sigma^2 = \phi(0) = N_0 W = N,$$

we thus have,

$$p(\vec{v}|N_0) = \left(\frac{1}{2\pi N}\right)^{Wt} e^{-1/2N \sum_{j=1}^{2Wt} v_j^2},$$

where \vec{v} represents the set of samples measured and corresponds to a noise reading of length t . Using the orthonormality of the base functions employed in the sampling theorem, we can show that

$$\sum_{j=1}^{2Wt} v_j^2 = 2W \int_0^t d\tau v^2(\tau),$$

which allows the wave form probability density to be written in a form that no longer makes any explicit reference to the sampling procedure:

$$p(V_0^t | NO) = k e^{-1/N_0 \int_0^t d\tau v^2(\tau)}$$

Using similar techniques, we can show that if the signal is also present in the received voltage history, the probability distribution of the resultant wave form is:

$$p(V_0^t | YES) = k e^{-1/N_0 \int_0^t d\tau (v(\tau) - s(\tau))^2},$$

and, therefore, the amount of information contained within the voltage history V_0^t regarding the presence of a signal wave form becomes:

$$\begin{aligned} & J[p(YES|V_0^0), p(YES|V_0^t)] \\ &= \log \frac{e^{-1/N_0 \int_0^t d\tau (v(\tau) - s(\tau))^2}}{p(YES|V_0^0) e^{-1/N_0 \int_0^t d\tau (v(\tau) - s(\tau))^2} + p(NO|V_0^0) e^{-1/N_0 \int_0^t d\tau v^2(\tau)}} \end{aligned} \quad (4)$$

To simplify this expression, a dimensionless energy parameter $\rho(t)$, is defined by the equation:

$$\rho^2(t) = \frac{2E}{N_0} ,$$

where:

$$E = \int_0^t d\tau s^2(\tau) ,$$

is the signal energy received during the interval of observation t . Equation (4) may then be written

$$\begin{aligned} J[p(\text{YES}|V_0^0), p(\text{YES}|V_0^t)] \\ = \log \frac{e^{2/N_0} \int_0^t d\tau v(\tau) s(\tau)}{p(\text{YES}|V_0^0) e^{2/N_0} \int_0^t d\tau v(\tau) s(\tau) + p(\text{NO}|V_0^0) e^{\rho^2(t)/2}} . \end{aligned} \quad (5)$$

Notice that the amount of information regarding the presence of a target signal depends upon the measured voltage history only through the correlation integral,

$$q(t) = \int_0^t d\tau v(\tau) s(\tau) .$$

Since this quantity will tend to be greatest when the received wave form actually contains the given target signal, the information regarding the signal's presence within the received wave form will also be greatest when the received voltage history contains the signal.

The form that the receiver should take in the case just considered follows from the preceeding theory. Specifically, equation (5) shows that in order to obtain the relevant information, the received wave form should be passed through a filter whose response to a unit impluse is

$$g(t) = \begin{cases} s(t-\tau) & 0 \leq \tau \leq t \\ 0 & \text{otherwise} \end{cases},$$

where, as before, t is the period of observation. Indeed, if the signal is completely known, the output of the filter at the end of the period of observation yields a voltage proportional to the correlation integral $q(t)$. Thereupon, the amount of information the observer has gathered follows from the simple non-singular transformation of the filter output indicated by equation (5). Finally, from equations (1) and (2), the observer's state of knowledge regarding the presence of the signal within the received voltage history is given by:

$$p(\text{YES}|V_0^t) = p(\text{YES}|V_0^0) e^{J[p(\text{YES}|V_0^0), p(\text{YES}|V_0^t)]}.$$

Needless to say, we can develop the theory along similar lines to encompass the calculation of the amount of information regarding any other parameter of

tactical value, and thus ascertain the form of the corresponding optimal receiver. Also, we may attempt to apply the approach we have taken above to a case with a somewhat more realistic description of the noise process, and it is to this latter problem that we turn our attention first.

THE CASE OF A STOCHASTIC SIGNAL

If applications of the framework described above to the problem of radar reception were our purpose, the correlation receiver would not be such a bad approximation. In sonar detection, however, the fundamental assumptions made in studying the deterministic signal case are not acceptable. The most important difficulty is due to the fact that the signal, when present, has a stochastic character, and the neglect of signal distortion due to the propagation medium is no longer a tenable position.

To illustrate the manner in which we might extend the theory to accommodate stochastic signals, we adopt a particular model (reference 12) of the signals and of the noise in which they are to be received, and attempt to evaluate the information that the received voltage history contains concerning the existence parameter. The signal to be detected is taken as a segment of a narrowband wave with a complex, random envelope $Z(\tau)$ and a carrier frequency Ω ,

$$s(\tau) = \text{Re } Z(\tau) e^{i\Omega\tau}.$$

Here, only the real part of the complex wave is to be taken as the signal wave form. Of the random envelope, we only know that its components are Gaussian random processes with given complex correlation functions

$$\phi(\tau_1, \tau_2) = \frac{1}{2} \langle Z(\tau_1) Z^*(\tau_2) \rangle ,$$

where the asterisk notation represents complex conjugation of the quantity it modifies. The input to the receiver system always contains Gaussian noise to which the signal, when it occurs, is added. We shall assume for the sake of computational simplicity, that the background noise is white, providing for the correlation function of the noise process

$$\frac{1}{2} \langle n(\tau_1) n^*(\tau_2) \rangle = N \frac{\sin 2\pi W(\tau_2 - \tau_1)}{2\pi W(\tau_2 - \tau_1)} .$$

Since the actual value of the noise bandwidth W does not appear in the result, we take the limit as this bandwidth approaches infinity and thus have a Dirac delta correlation function

$$\langle n(\tau_1) n^*(\tau_2) \rangle = N_0 \delta(\tau_2 - \tau_1) .$$

Therefore, since the signal and the noise are statistically independent, the complex correlation function of the received voltage envelope when both are present is

$$\langle v(\tau_1) v^*(\tau_2) \rangle = \phi(\tau_1, \tau_2) + N_0 \delta(\tau_2 - \tau_1) . \quad (6)$$

To proceed, we might use, as we have before, the values $v(j/2W)$ of the voltage history at propitiously chosen time points. However, these values are no longer statistically independent of each other; and because of the form of their joint probability density function, the passage to the continuum is somewhat more difficult to analyze than it was in the example described before. It will be simpler, rather, to find a set of quantities v_j that are uncorrelated but can be generated by linear operations on the voltage wave form. These quantities will be the coefficients of an expansion of the received voltage $v(\tau)$ in a particular kind of Fourier series (reference 13).

We accomplish such by writing for the received voltage,

$$v(\tau) = \sum_j v_j f_j(\tau) \quad , \quad (7)$$

where the set of functions $f_j(\tau)$ is orthonormal with respect to the time interval $0 < \tau < t$,

$$\int_0^t d\tau f_j^*(\tau) f_k(\tau) = \delta_{jk} \quad .$$

Then, it is not difficult to show that the expansion coefficients in equation (7) can be computed from

$$v_j = \int_0^t d\tau v(\tau) f_j^*(\tau) \quad .$$

The expansion coefficients are therefore linearly related to the input voltage $v(\tau)$, and the j th one could be generated by passing the voltage through a linear filter matched to a signal of the form $f_j(\tau)$.

Since the voltage history is random, so are the expansion coefficients v_j . Their covariances are given by the following equation,

$$\langle v_j v_k^* \rangle = \int_0^t d\tau_1 d\tau_2 f_j^*(\tau_1) f_k(\tau_2) \langle v(\tau_1) v^*(\tau_2) \rangle ,$$

and using equation (6),

$$\langle v_j v_k^* \rangle = N_0 \delta_{jk} + \int_0^t d\tau_1 d\tau_2 \phi(\tau_1, \tau_2) f_j^*(\tau_1) f_k(\tau_2) . \quad (8)$$

Our goal will be achieved if we can find a set of orthonormal functions such that this expression vanishes whenever j and k are different. This will be so, if for each function $f_j(\tau)$ separately we require

$$\int_0^t d\tau f_j(\tau) \phi(\tau, \tau') = \lambda_j f_j(\tau') , \quad (0 < \tau' < t) \quad (9)$$

where the constants λ_j have yet to be determined. Substituting equation (9) into the covariance equation (8), we find

$$\langle v_j v_k^* \rangle = (\lambda_j + N_0) \delta_{jk} ,$$

so that the expansion coefficients are indeed uncorrelated, and their variances are simply related to the eigen values (reference 14) λ_j of the integral equation (9),

$$\langle |v_j|^2 \rangle = \lambda_j + N_0.$$

Hence, the joint probability density of the entire set of coefficients is again the product of each separate distribution and, since the coefficients determine the wave form, this product gives the probability density for the wave form itself. But, because the voltage coefficients v_j are formed by linear operations on the Gaussian process $v(\tau)$, they must be Gaussian random variables, and we finally obtain,

$$p(V_0^t | \text{YES}) = \lim_{M \rightarrow \infty} \prod_{j=1}^M \left(\frac{1}{2\pi(\lambda_j + N_0)} \right) e^{-\sum_{j=1}^M |v_j|^2 / 2(\lambda_j + N_0)}.$$

Furthermore, if noise alone were present, the probability density for the voltage history would trivially be given by

$$p(V_0^t | \text{NO}) = \lim_{M \rightarrow \infty} \left(\frac{1}{2\pi N_0} \right)^M e^{-\frac{1}{2N_0} \sum_{j=1}^M |v_j|^2},$$

and, therefore, the amount of information concerning the presence of the target signal within the voltage history already viewed by the observer becomes

$$J[p(\text{YES}|V_0^0), p(\text{YES}|V_0^t)]$$

$$= \log \frac{\prod_{j=1}^{\infty} \left(\frac{1}{1+\lambda_j/N_0} \right) e^{1/2N_0 \sum_{j=1}^{\infty} \lambda_j |v_j|^2 / (\lambda_j + N_0)}}{p(\text{YES}|V_0^0) \prod_{j=1}^{\infty} \left(\frac{1}{1+\lambda_j/N_0} \right) e^{1/2N_0 \sum_{j=1}^{\infty} \lambda_j |v_j|^2 / (\lambda_j + N_0)} + p(\text{NO}|V_0^0)} \quad (10)$$

provided

$$\int_0^t d\tau_1 d\tau_2 |\phi(\tau_1, \tau_2)|^2 < \infty$$

to ensure that the limit as the number of coefficients tends to infinity does indeed exist (reference 15).

Before we can attempt to determine the design of the optimum detector suggested by equation (10), we must cast this equation into a more suitable form.

Thus, employing the equation defining the voltage coefficient v_j ,

$$\frac{1}{2} \sum_{j=1}^{\infty} \lambda_j |v_j|^2 / (\lambda_j + N_0) = 1/2N_0 \int_0^t d\tau_1 d\tau_2 v^*(\tau_1) h(\tau_1, \tau_2, 1/N) v(\tau_2) ,$$

where we have defined

$$h(\tau_1, \tau_2; u) \equiv \sum_{j=1}^{\infty} f_j(\tau_1) f_j^*(\tau_2) \lambda_j / (1 + \lambda_j u) \quad (11)$$

Furthermore, the infinite product in equation (10),

$$D(1/N_0) \equiv \prod_{j=1}^{\infty} (1 + \lambda_j / N_0) ,$$

can also be evaluated in terms of the function defined in equation (11). Indeed, taking the logarithm of $D(1/N_0)$, we have

$$\ln D(1/N_0) = \sum_{j=1}^{\infty} \ln(1 + \lambda_j/N_0) = \sum_{j=1}^{\infty} \int_0^{1/N_0} du \frac{\lambda_j}{1 + \lambda_j u} . \quad (12)$$

But, setting $\tau_1 = \tau_2$ in equation (11) and integrating from 0 to t , we obtain, upon using the normalization condition for the f_j functions, the integral on the right-hand side of equation (12). Therefore (reference 16),

$$D(1/N_0) = e^{\int_0^{1/N_0} du \int_0^t d\tau h(\tau, \tau; u)} ,$$

and

$$\begin{aligned} & J[p(\text{YES}|V_0^0), p(\text{YES}|V_0^t)] \\ &= \log \frac{e^{\frac{1}{2N_0^2} \int_0^t d\tau_1 d\tau_2 v^*(\tau_1) h(\tau_1, \tau_2; 1/N_0) v(\tau_2)}}{p(\text{YES}|V_0^0) e^{\frac{1}{2N_0^2} \int_0^t d\tau_1 d\tau_2 v^*(\tau_1) h(\tau_1, \tau_2; 1/N_0) v(\tau_2)} + P(\text{NO}|V_0^0) D(1/N_0)} \end{aligned} \quad (13)$$

indicating that a knowledge of $h(\tau_1, \tau_2; u)$ is sufficient to provide the information contained within the given sample path V_0^t . If we multiply both sides of equation (11) by $u\phi(\tau, \tau_1)$ and integrate over the range $0 < \tau_1 < \tau$, we get, using the integral equation (9)

$$u \int_0^t d\tau_1 \phi(\tau_1, \tau_1) h(\tau_1, \tau_2; u) = \sum_{j=1}^{\infty} f_j(\tau) f_j^*(\tau_2) \lambda_j^2 / (1 + \lambda_j / u)$$

which, upon addition to equation (11), provides

$$h(\tau_1, \tau_2; u) + u \int_0^t d\tau \phi(\tau_1, \tau) h(\tau, \tau_2; u) = \sum_{j=1}^{\infty} \lambda_j f_j(\tau_1) f_j^*(\tau_2) .$$

Finally, since

$$\phi(\tau_1, \tau_2) = \sum_{j=1}^{\infty} \lambda_j f_j(\tau_1) f_j^*(\tau_2) ,$$

as can be easily shown, we find that $h(\tau_1, \tau_2; u)$ is the solution of the following integral equation:

$$h(\tau_1, \tau_2; u) + u \int_0^t d\tau \phi(\tau_1, \tau) h(\tau, \tau_2; u) = \phi(\tau_1, \tau_2) , \quad 0 < (\tau_1, \tau_2) < t . \quad (14)$$

We are now ready to specify the optimum receiver implied by equation (13) in terms of the solution to the integral equation (14). To do so, let

$$W(\tau_1) = \int_0^1 d\tau_2 h(\tau_1, \tau_2; 1/N) v(\tau_2) ,$$

and thus write the elements in equation (13) as

$$1/2N \int_0^t d\tau_1 d\tau_2 v^*(\tau_1) h(\tau_1, \tau_2; 1/N_0) v(\tau_2) = 1/N \int_0^t d\tau_1 v^*(\tau_1) W(\tau_1) .$$

For each value of τ_1 , $W(\tau_1)$ is a weighted average of the input voltage history $v(\tau)$ that has been viewed previously to the time τ_1 . It can be found by passing the voltage received through a time-variable linear filter having a narrowband impulse response

$$g_1(\tau) = \begin{cases} h(\tau_1, \tau_1 - \tau, 1/N) & 0 < \tau < \tau_1 \\ 0 & \text{otherwise} \end{cases} .$$

The output of this filter at time τ is $\text{Re } W(\tau)e^{i\Omega\tau}$. Therefore, if we now multiply this at each instant by the input $\text{Re } v(\tau)e^{i\Omega\tau}$ and thereupon integrate the result with a low-frequency filter, the output at time t is essentially equal to the exponent in equation (13), and this system evaluates the amount of information contained within the received voltage history regarding the presence of a Gaussian narrowband signal.

PASSIVE RANGING

We have applied the sample-path information concept of Woodward & Davies to the estimation of the existence parameter under varied assumptions about the underlying statistical properties of the voltage process. There exists, however, no objection to extending the application of this concept to encompass the calculation of the amount of information regarding any other parameter of tactical value. In particular, one wants to estimate the position of the target with

respect to some specified system of reference, by evaluating

$$J[p(\vec{r}|V_0^0), p(\vec{r}|V_0^t)] = \log \frac{U(\vec{r}|V_0^t)}{U(\vec{r}|V_0^0)}$$

where the observer's state of knowledge about the target's position \vec{r} is given by the corresponding rule of Bayes,

$$U(\vec{r}|V_0^t) = \frac{U(\vec{r}|V_0^0) p(V_0^t|\vec{r})}{\int d\vec{\rho} U(\vec{\rho}|V_0^0) p(V_0^t|\vec{\rho})}.$$

To carry out such a program, one must possess a reasonable model of the acoustic process, capable to provide the range and bearing dependence of the conditional probability density $p(V_0^t|\vec{r})$. We are not prepared at this time to pursue this problem beyond the feasibility remarks above, but will return to it upon completion of a few, somewhat simpler, investigations that naturally suggest themselves within an information-theoretic approach to passive acoustic detection. It is important to notice, however, that if one is willing to view passive acoustic search as an information-gathering process, there exists an entirely natural and conceptually straightforward way of posing the problem of ranging to the target. In fact, estimating the target position is in no way different from estimating the target's existence parameters. As we have already pointed out before, this highly desirable feature of the information-theoretic approach to detection is entirely lacking in the conventional theory.

Indeed, modern passive detection apparatus is conspicuously incapable of providing ranging information on a previously detected target, although that information was definitely present in the voltage history used in evaluating the existence parameter.

SECTION IV

THE CONVENTIONAL APPROACH TO DETECTION

The preceeding development has been fundamentally predicated upon the suggestion that searching for a submarine is primarily a problem in information-gathering. Correspondingly, we have found it quite natural to represent the evolution of the search process by the time-dependence of the observer's state of knowledge concerning the complete set of tactically relevant parameters. In general, the attendant body of theory appeared not only to allow the evaluation of the searcher's state of knowledge along some voltage sample path, but also suggested the optimal electronic procedure to be used in gathering the desired information.

Notwithstanding the general conceptual appeal of this approach to the acoustic search problem, history has apparently chosen to ignore it. To be sure, many of the concepts presented here have, in some form or another, been available within the boundaries of conventional theory; but, to achieve from this vantage point complete contact with the classical approach, we must make drastic and often quite untenable assumptions the impact of which, while hard to evaluate in the absence of the correct result, is nevertheless bound to be significant.

Within the conventional theory, a good deal of relevant information actually present in the voltage history is disregarded in that the complete set of tactical parameters is generally reduced to the relatively uninteresting existence parameter.

Despite the fact that the observer is ultimately interested in learning the position of his target, it appears that the only question the observer is instructed to ask of the voltage history is whether or not a target is actually present in the surrounding ocean. To describe his knowledge concerning the existence parameter, the conventional theory insists (reference 16) that the observer make objective use of the voltage data he has collected. It is therefore not acceptable to refer his current state of knowledge to an a priori one, unless such is expressible as an objective frequency statement about the parameter values at some initial time. Correspondingly, the conventional analyst, instead of using his measurement to evaluate the amount of information concerning the presence of the submarine, would use his measurement to evaluate a related, but not equivalent, quantity that makes no reference whatsoever to the observer's a priori state of knowledge. This quantity, generally referred to as the likelihood ratio of the measured sample, no longer admits of any simple interpretation in terms of concepts relevant to the original search question and, therefore, the measured value of the likelihood ratio does not naturally relate to any description the observer might consider reasonable for his knowledge concerning the existence parameter.

To learn about the presence of a target from his measurement, the conventional observer must arbitrarily correspond the value of the likelihood ratio to the possible values of the existence parameter. In general, he assumes that the target is present if, and only if, the likelihood ratio has exceeded some pre-established value, the magnitude of which is usually so determined that noise

alone would seldom cause the measured value to go above it (reference 17). This stochastic, threshold-crossing event will be called detection. Needless to say, the conventional way to establish the presence of a submarine in the ocean implies the destruction of some finite amount of information. It is therefore to be expected that a processor operating in accordance with the conventional framework will not perform the task of determining the presence of targets as well as it might have without the limitation imposed upon it by the information thresholding process.

To quantify this performance, the conventional theory evaluates the probability that a detection event will have occurred if a target were actually present, and calls this measure of effectiveness the probability of detection. We may illustrate these concepts by returning to the simple example developed in our discussion of a deterministic signal where a known, undistorted signal is mixed into stationary, white Gaussian noise. Under these circumstances, the likelihood ratio of the sample \vec{V} ,

$$\Lambda(\vec{V}) \equiv \frac{p(\vec{V} | H_1 S)}{p(\vec{V} | H_0)} ,$$

is a monotonically increasing function of the correlation integral

$$q(t) = \int_0^t d\tau \, v(\tau) s(\tau) ,$$

where t represents the time-length of sample viewed, and a detection event is called if $q(t)$ exceeds a predetermined threshold q_0 .

In principle, there are a variety of ways in which the threshold might meaningfully be chosen, but the most popular practice is to establish a q_0 that will maximize the detection probability subject to a specified, operationally acceptable value for the probability of false alarm. It is not difficult to determine the probability density function of the correlation integral and, hence, the detection probability within this model. In fact, since the measured voltage is assumed to be a Gaussian process, the correlation integral for any fixed value of t is a Gaussian random variable of mean value

$$\langle q(t) \rangle_{\text{YES}} = \int_0^t d\tau \langle v(\tau) \rangle_{\text{YES}} \quad s(\tau) = \int_0^t d\tau s^2(\tau) ,$$

if the target is indeed present, and of corresponding variance

$$\begin{aligned} \langle (q(t) - \langle q(t) \rangle)^2 \rangle_{\text{YES}} &= \int_0^t d\tau_1 \int_0^t d\tau_2 \langle n(\tau_1) n(\tau_2) \rangle s(\tau_1) s(\tau_2) \\ &= \frac{1}{2} N_0 \int_0^t d\tau_1 \int_0^t d\tau_2 \delta(\tau_2 - \tau_1) s(\tau_1) s(\tau_2) \\ &= \frac{1}{2} N_0 \int_0^t d\tau s^2(\tau) . \end{aligned}$$

Therefore,

$$p(q(t); \text{YES}) = \left(\frac{1}{\pi N_0 E} \right)^{1/2} e^{-\frac{(q(t) - E)^2}{N_0 E}} ,$$

and the probability of detection,

$$Q_d \equiv \int_{q_0}^{\infty} dq p(q|YES)$$

will be given by

$$Q_d = \frac{1}{2} \left[1 - \operatorname{erf} \left(\frac{q_0 - E}{\sqrt{N_0 E}} \right) \right] ,$$

where E represents the energy dissipated during the observation interval in a resistor of unit resistance when a voltage $s(\tau)$ is put across it, and the threshold value q_0 is related to the false alarm probability,

$$Q_{FA} \equiv \int_{q_0}^{\infty} dq p(q|NO) ,$$

by the equation

$$Q_{FA} = \frac{1}{2} \left[1 - \operatorname{erf} \left(\frac{q_0}{\sqrt{N_0 E}} \right) \right] .$$

Introducing the signal-to-noise ratio

$$\rho_s^2(t) = 2E/N_0 ,$$

we may write the detection probability as

$$Q_d(\sigma) = \frac{1}{2} \left[1 - \operatorname{erf} \left(\frac{D - \rho(t)}{\sqrt{2}} \right) \right], \quad (15)$$

where D is related to the preassigned false alarm probability through

$$Q_{FA} = \frac{1}{2} \left[1 - \operatorname{erf} \left(\frac{D}{\sqrt{2}} \right) \right]. \quad (16)$$

It is to be noted that from equation (18) the probability of detection for a given false alarm probability is a function of the signal-to-noise ratio alone. This property continues to hold true for an entire class of models, but such need not be the case in general.

We shall close this section of our review with yet another example of how the conventional theory ascertains the presence of acoustic targets. Although this model has little, if any, intrinsic claim to attention, it has received sufficiently widespread application to warrant inclusion here. It assumes (reference 11) that the signal is represented by a white, Gaussian process of mean power density S_0 , and that, when present, this signal is imbedded in white, Gaussian noise of power density N_0 . Therefore,

$$p(\vec{v}|N0) = \left(\frac{1}{2\pi N_0} \right)^{Wt} e^{-1/2N_0 \sum_{j=1}^{2Wt} v_j^2}$$

$$p(\vec{v}|YES) = \left(\frac{1}{2\pi(N_0+S_0)} \right)^{Wt} e^{-1/2(N_0+S_0) \sum_{j=1}^{2Wt} v_j^2}$$

and the likelihood ratio is a monotonic function of the statistic

$$G \equiv 1/N_0 \sum_{j=1}^{2Wt} v_j^2 = 1/2N_0 \int_0^t d\tau v^2(\tau) .$$

Under the null hypothesis, G is χ^2 -distributed

$$p(G|N0) = \frac{1}{2} \frac{1}{\Gamma(Wt)} e^{-G/2} G^{Wt-1}$$

and, correspondingly

$$Q_{FA} = \int_{G_U}^{\infty} dG p(G|N0) ,$$

Under the alternate hypothesis, G is no longer χ^2 distributed, but the related statistic

$$\hat{G} \equiv \frac{N_0}{N_0 + S_0} G$$

still is. Hence,

$$p(\hat{G} | \text{YES}) = \frac{1}{2^{\frac{Wt}{2}} \Gamma(\frac{Wt}{2})} e^{-\hat{G}/2} \hat{G}^{Wt/2-1} \frac{d\hat{G}}{dG}$$

and,

$$Q_d = \int_{G_0}^{\infty} dG p(G | \text{YES}) .$$

If a variate is χ^2 distributed with degrees of freedom n larger than 30, the square root of that variate is approximately normal with mean $\sqrt{2n-1}$ and unit variance. Therefore, as soon as $2WT \gg 30$,

$$Q_{FA} = \frac{1}{\sqrt{2\pi}} \int_{\sqrt{2G_0}}^{\infty} d\xi e^{-(\xi - \sqrt{4Wt-1})^2/2}$$

$$Q_d = \frac{1}{\sqrt{2\pi}} \int_{\sqrt{\frac{2N_0}{N_0+S_0} G_0}}^{\infty} d\xi e^{-(\xi - \sqrt{4Wt-1})^2/2} .$$

Performing the ξ -equation, we have:

$$Q_{FA} = \frac{1}{2} \left[1 - \operatorname{erf} \left(\frac{\sqrt{2G_0} - \sqrt{4Wt-1}}{\sqrt{2}} \right) \right],$$

$$Q_d = \frac{1}{2} \left[1 - \operatorname{erf} \left(\frac{\sqrt{\frac{2N_0}{N_0+S_0}} G_0 - \sqrt{4Wt-1}}{\sqrt{2}} \right) \right].$$

Finally, if $S_0/N_0 \ll 1$,

$$Q_{FA} = \frac{1}{2} \left[1 - \operatorname{erf} \left(\frac{D}{\sqrt{2}} \right) \right]$$

$$Q_d = \frac{1}{2} \left[1 - \operatorname{erf} \left(\frac{D - \sqrt{Wt} S_0/N_0}{\sqrt{2}} \right) \right]$$

which corresponds to equations (15) and (16) if we replace

$$\rho^2(t) \Rightarrow \frac{Wt S_0^2}{N_0^2} . \quad (17)$$

Naturally, the optimum receiver against such a signal is an energy integrator representing at the output terminals the statistic

$$G(t) = \int_0^t dt v^2(t) .$$

THE SIGNAL EXCESS MODEL OF DETECTION

The examples briefly studied above appear quite useful for illustrating the concepts involved in the conventional theory, but can hardly be considered representative of the real detection process for, as previously indicated, the signal is fundamentally stochastic, with properties that are strongly related to the physical phenomena occurring in the ocean. A natural attempt at accommodating this randomness would be to reformulate in the conventional language the narrowband stochastic signal model with white Gaussian noise, or whatever other model might be suggested by an analysis of ocean-fluctuation phenomena (reference 15).

Practitioners of the conventional approach have found it, however, more expedient to address the question of randomness in the signal from an entirely different point of view. According to this "signal excess model" of acoustic detection, it is the randomness in the signal-to-noise ratio process rather than the randomness in the signal and noise processes that must be related to the parameters characterizing the ocean environment. The conventional philosophy would thus instruct us to evaluate the mean likelihood ratio

$$\langle \Lambda \rangle = \int d\rho P(\rho) \Lambda(\rho) ,$$

where $P(\rho)$ represents the probability density function for the signal-to-noise ratio, and subsequently to compare it to a preassigned threshold value Λ_0 . The probability for a detection event to occur can therefore be related to the

measure of effectiveness for a processor designed to perform against a given, non-random, signal-to-noise ratio,

$$P_d = \int_0^\infty d\rho \, P(\rho) \, \Pr(\Lambda(\rho) \geq \Lambda_0) .$$

It is important to notice, however, that the probability for $\Lambda(\rho) \geq \Lambda_0$ is not strictly equal to $Q_d(\rho)$ as evaluated in the previous section, because the threshold setting is now chosen to maximize P_d at an acceptable level of false alarm probability rather than $Q_d(\rho)$ itself. With that understanding in mind, we write

$$P_d = \int_0^\infty d\rho \, P(\rho) \, Q_d(\rho) ,$$

and the observer has agreed to be satisfied with a threshold-setting strategy that performs well against an ensemble of possible values of the parameter distributed according to $P(\rho)$.

Rather than use any of the models for $Q_d(\rho)$ discussed previously, it is the custom to take

$$Q_d(\rho) = \begin{cases} 1 & \rho \geq \rho_{1/2} \\ 0 & \text{otherwise} \end{cases} ,$$

where $\rho_{1/2}$ is that signal-to-noise ratio at which the probability of detection $Q_d(\rho)$ becomes equal to $1/2$. Consequently,

$$P_d = \int_1^{\infty} d\xi P(\xi) ,$$

for $\xi \equiv \rho/\rho_{1/2}$. It is further customary to introduce the signal excess variable

$$SE \equiv 10 \log \xi$$

and write

$$P_d = \int_0^{\infty} dSE P(SE) . \quad (18)$$

Equation (18) is a succinct statement of the signal excess detection model.

The signal excess variable is generally written as

$$SE = 10 \log \rho - RD$$

where

$$RD = 10 \log \rho_{1/2}$$

is called the recognition differential (reference 19). If we were to choose as it is often done in practice, the white Gaussian signal model to describe $Q_d(\rho)$, then the recognition differential would be given by

$$RD = 5 \log \frac{DW}{t}$$

with the preassigned false alarm probability related to D via equation (16). The first term in the signal excess equation can be evaluated from the sonar equation.

THE SIGNAL EXCESS PROCESS

The fundamental ingredient to the signal excess model of equation (18) is the probability density function, $P(SE)$, describing the statistical properties of the signal excess process, $SE(t)$. Although this function should be specified as soon as some acceptable model of the ocean acoustic fluctuations is made available, the complexity of the problem and a variety of other circumstances have encouraged the practice whereby, guided by the principle of simplicity, we choose a mathematical model for $P(SE)$ with a few undetermined parameters and subsequently proceed to match the available degrees of freedom to the data.

The Uncorrelated Gaussian Model

A simple and hence commonly employed model chosen for $P(SE)$ is a Gaussian model with mean value $\langle SE(t) \rangle$ as provided by the sonar equation

$$\langle SE(t) \rangle = \langle FGM \rangle - \langle PL(t) \rangle$$

and with adjustable variance,

$$P(SE) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(SE - \langle SE(t) \rangle)^2 / 2\sigma^2} .$$

In this model, the signal excess distribution at time t is independent of the previously measured values for the signal excess and, hence, the random process $SE(t)$ is uncorrelated. The detection probability is trivially evaluated from equation (21) to be

$$P_d = \frac{1}{2} \left[1 - \operatorname{erf} \left(- \frac{\langle SE(t) \rangle}{\sigma \sqrt{2}} \right) \right] .$$

Notice that $\langle SE(t) \rangle = 0$ corresponds to a probability of detection of $\frac{1}{2}$; this accounts for the widespread use of the range at which the propagation loss curve crosses the mean figure-of-merit line as a measure of effectiveness for sonar performance.

Values for the variance can be obtained by fitting the theory to specified sets of exercise data, but due to the general paucity of reliable data, the choice of σ is mostly a matter of taste.

From a tactical viewpoint, the instantaneous probability of detection, $P_d(t)$, is of limited value. A widely accepted measure of tactical effectiveness is the cumulative probability of detection (reference 20) through some time interval $(0, t)$,

$$F(t) \equiv \text{Prob} [SE(\tau) \geq 0 \text{ , for some } \tau \in (0,t)]$$

$$= 1 - \text{Prob} [SE(\tau) < 0 \text{ , for all } \tau \in (0,t)] \text{ .}$$

In the uncorrelated Gaussian model discussed here, the cumulative probability of detection can be evaluated as

$$F_M = 1 - \prod_{j=1}^M (1 - P_d(t_j))$$

where the time interval $(0,t)$ has been partitioned into M equal pieces, and $P_d(t_j)$ is the instantaneous probability of detection at the beginning of the j th interval. Naturally, for $P_d(t)$ a continuous function of time,

$$\lim_{M \rightarrow \infty} F_M = 1$$

$$M \rightarrow \infty$$

$$\Delta t \rightarrow 0$$

$$M \Delta t = t$$

The $\lambda - \sigma$ Jumo Model

The signal excess process is again assumed to have a normally distributed marginal with mean $\langle SE(t) \rangle$ and variance σ^2 , but the signal excess values at different time instances are no longer statistically independent (reference 21). In fact, the process is fully correlated for intervals of time of random duration.

distributed Poisson with some freely adjustable intensity λ , and fully uncorrelated otherwise.

It is not difficult to evaluate the correlation function for this model of signal excess variability. Indeed (reference 21), let us consider the statistical dependence between the signal excess values at t and $t + \tau$, as measured by the correlation function

$$b(t, t+\tau) \equiv \langle SE(t) SE(t+\tau) \rangle - \langle SE(t) \rangle \langle SE(t+\tau) \rangle .$$

With probability $e^{-\lambda\tau}$, there will have been no jump during the time interval τ and, hence, with that probability

$$SE(t+\tau) = SE(t) .$$

With probability $(1 - e^{-\lambda\tau})$ at least one jump has occurred during the time interval τ , thus totally decorrelating the signal excess process

$$\text{Prob}(SE(t) , SE(t+\tau)) = \text{Prob}(SE(t)) \text{Prob}(SE(t+\tau)) .$$

Therefore,

$$\langle SE(t) SE(t+\tau) \rangle = e^{-\lambda\tau} \int_{-\infty}^{\infty} dx x^2 \text{Prob}(x)$$

$$+ (1-e^{-\lambda\tau}) \int_{-\infty}^{\infty} dx dy xy \text{Prob}(x) \text{Prob}(y)$$

$$= e^{-\lambda\tau} (\sigma^2 + \langle SE(t) \rangle^2)$$

$$+ (1-e^{-\lambda\tau}) \langle SE(t) \rangle \langle SE(t+\tau) \rangle ,$$

and,

$$b(t, t+\tau) = \sigma^2 e^{-\lambda\tau} + \langle SE(t) \rangle (\langle SE(t+\tau) \rangle - \langle SE(t) \rangle) e^{-\lambda\tau} .$$

If the mean signal excess is time-independent, the $\lambda - \sigma$ jump model correlation function becomes

$$b(\tau) = \sigma^2 e^{-\lambda\tau} .$$

By definition, the instantaneous probability of detection is the same here as it was in the uncorrelated model; the cumulative probability of detection, however, is different and no longer trivial. Thus, if the mean signal excess is constant in time,

$$F(t) = 1 - \text{Prob} \left(SE(\tau) < 0, \text{ for all } \tau \in (0, t) \right)$$

$$= 1 - \sum_{j=0}^{\infty} \text{Prob} \left(SE(\tau) < 0, \text{ for all } \tau \in (0, t) \mid j \text{ jumps} \right) \text{Prob} (j \text{ jumps})$$

$$= 1 - \sum_{j=0}^{\infty} (1 - p_d)^{j+1} \frac{(\lambda t)^j}{j!} e^{-\lambda t}$$

so that,

$$F(t) = 1 - (1 - p_d) e^{-\lambda p_d t}.$$

From this equation it is seen that the mean time to detection,

$$\mu = \int_0^{\infty} t \, dF(t)$$

is given by

$$\mu = \frac{1 - p_d}{\lambda p_d}.$$

One can similarly show that if contact is held at time $t = 0$, i.e., $SE(0) \geq 0$, then the probability distribution of the time θ at which contact is lost for the first time is given by

$$\text{Prob} (\theta \leq t) = 1 - p_d e^{-\lambda(1-p_d)t}.$$

The next level of complexity of known results concerns the case when the mean signal excess is time-dependent and unimodal, that is $SE(t)$ is non-decreasing on an interval $(0, t_0)$ and non-increasing on (t_0, ∞) . For this case, the cumulative probability of detection can again be computed exactly (references 21 and 22),

$$F(t) = \begin{cases} 1 - (1-p_d(t)) e^{-\lambda \int_0^t d\tau p_d(\tau)} & , 0 \leq t \leq t_0 \\ 1 - (1-p_d(t_0)) e^{-\lambda \int_0^t d\tau p_d(\tau)} & , t \geq t_0 \end{cases}.$$

For arbitrary, non-unimodal, mean signal excess histories, there exists no closed-form solution to the problem of evaluating the cumulative probability of detection, but we can approximate it to within any desired degree of accuracy (reference 23).

The Gauss-Markov Model

In this model, the signal excess process is assumed to be a Markov process for which all finite dimensional distributions are normal. Just like the $\lambda - \sigma$

jump model, the Gauss-Markov model describes a correlated signal excess process with correlation function (reference 24),

$$b(\tau) = \sigma^2 e^{-\lambda|\tau|}$$

However, unlike the jump process, there are no useful results to be obtained in closed form. The principal means for computing the cumulative probability of detection are approximation methods extensively covered elsewhere (reference 22) and, therefore, we shall not pursue this point any further. We will note, however, that application of this model to some specified circumstances seems to indicate that the cumulative probability of detection for the Gauss-Markov process is larger than the value obtained when the $\lambda - \sigma$ jump model with the same parameters is used under the same circumstances (reference 25). Furthermore, one shows (reference 26) that the Gauss-Markov process allows, on the average, much shorter contact times than the $\lambda - \sigma$ jump process. In fact, for large positive values of $\langle SE \rangle$, the mean time to loss of contact within a Gauss-Markov model,

$$\mu_G = \frac{\sqrt{\pi}}{\lambda} \sum_{j=1}^{\infty} \frac{(\langle SE \rangle / \sigma)^j}{2^{j/2} j \Gamma\left(\frac{j+1}{2}\right)}$$

becomes

$$\mu_J \xrightarrow{SE \rightarrow \infty} \mu_G \left(\frac{SE}{\sigma} \right)^2$$

where μ_j represents the mean time to loss of contact within the $\lambda - \sigma$ jump model.

The Log-Rice Model

All stochastic processes discussed above have been so designed as to have a normal marginal probability density function $P(SE)$ since a large body of data seems to indicate that such is indeed the case. Recent studies (reference 27) of the multipath arrival mechanism for randomness in the received signal seem to indicate, however, that under the appropriate circumstances, the signal excess process is well represented by (reference 28)

$$SE(t) = \log \left[\left(P + x(t) \right)^2 + y^2(t) \right]$$

where P is an arbitrary non-negative constant and $x(t)$, $y(t)$ are stationary Gauss-Markov processes. Although not much work has been done on this model as yet, one can show (reference 29) that the correlation function for the process is given by,

$$\frac{1}{2} B(\tau) = \int_0^{\frac{1}{2}} \frac{du}{u} e^{-Pu} \ln \frac{1}{1 - 2\rho u}$$

$$= \frac{1/(1+\rho)}{\frac{1}{2}} \int_{\frac{1}{2}}^{\frac{1}{2}} \frac{du}{u} e^{-Pu^2} \ln \frac{1}{1 - 2\rho u} = \frac{1/(1+\rho)}{\frac{1}{2}} \int_{\frac{1}{2}}^{\frac{1}{2}} \frac{du}{u} e^{-Pu^2} \ln \frac{1}{2u - 1}$$

where $\rho = e^{-\lambda|\tau|}$ represents the correlation coefficient for the underlying Gauss-Markov process and where $\sigma = 1$.

SECTION V CONCLUSIONS

We have argued that acoustic apparatus should be employed for collecting information about a complete set of tactically relevant parameters, rather than as decision-making devices. Within such a framework, information that is generally disregarded by the conventional procedure becomes available and can be actively used in whatever decision process one wants to adjoin to the information-gathering stage of the search game. This additional amount of information requires, however, that one possess a physical description of underwater acoustics detailed enough to provide the probability distribution for the acoustic disturbance process in terms of all tactically relevant parameters. The conventional theory is far less demanding on this point for it only requires descriptions of the acoustic fluctuation process in terms of ad hoc parameters such as σ and λ to be fitted to the available data. In fact, it is precisely because these parameters are not directly related to tactically relevant descriptions, that the conventional approach is not capable of obtaining the entire amount of information available in the voltage process.

It is not difficult, in principle at least, to show that the information-theoretic approach to acoustic search does indeed lead to advantages over the conventional framework. To do so, consider, for instance, that the voltage sample path V_0^t , has been drawn from an ensemble of paths characterized by $\epsilon = \text{YES}$. An observer that subscribes to the information-theoretic approach

to passive acoustic detection would use his processor to map this voltage history into a number describing his state of knowledge concerning, say $\epsilon = \text{YES}$, in accordance with equation (1),

$$p(\text{YES}|V_0^t) = \frac{p(\text{YES}|V_0^0) p(V_0^t|\text{YES})}{p(\text{YES}|V_0^0) p(V_0^t|\text{YES}) + p(\text{NO}|V_0^0) p(V_0^t|\text{NO})}.$$

As the sample path V_0^t grows in length, the observer's average state of knowledge $\langle p(\text{YES}|V_0^t) \rangle$ approaches certainty and the variance of his sample-path state of knowledge about this mean approaches zero. Therefore, by Chebyshev's inequality,

$$\text{Prob} \left[|p(\text{YES}|V_0^t) - \langle p(\text{YES}|V_0^t) \rangle| \leq \epsilon \right] \geq 1 - \frac{\text{Var}(p(\text{YES}|V_0^t))}{\epsilon}$$

the probability that the sample-path state of knowledge $p(\text{YES}|V_0^t)$ approach certainty with increasing viewing time gets arbitrarily close to unity. The conventional observer, on the other hand, will employ his processor to map the voltage sample-path into a succession of YESes and NOs by automatically thresholding a running-window statistic. Thereupon, depending upon the relative frequency of YESes in the succession, he will subjectively translate this succession into an ultimate decision regarding the presence of the target.

To make the comparison between the 2 observers meaningful, it seems reasonable to assume that the conventional observer will use the rule of Bayes to translate the succession of YESes and NOs he has available into a state

of knowledge concerning the existence parameter. Thus,

$$p(\text{YES} | D_0^t) = \frac{p(\text{YES} | D_0^0) p(D_0^t | \text{YES})}{p(\text{YES} | D_0^0) p(D_0^t | \text{YES}) + p(\text{NO} | D_0^0) p(D_0^t | \text{NO})}$$

where D_0^t represents a sample-path succession of ones and zeroes corresponding to the succession of YESes and NOs generated by automatic thresholding, and, for a running window of size W ,

$$p(\vec{D} | \text{YES}) = \int d\vec{v} p(\vec{v} | \text{YES}) \prod_{j=1}^{n-W} \theta \left[(-1)^{D_j} \left(G_0 - \sum_{k=j-W}^j v_k^2 \right) \right]. \quad (19)$$

is the probability of a discrete realization \vec{D} of the sample-path D_0^t .

It is not our intention to accomplish the numerical comparison between these two processing schemes at the present time, but it should be obvious by now that the only difference between them lies in the replacement of $p(V_0^t | \epsilon)$ with $p(D_0^t | \epsilon)$. As equation (19) indicates, the conventional processor performs an averaging operation over all those voltage sample-path V_0^t that produce the same sample-sequence D_0^t before evaluating the state of knowledge $p(\text{YES} | D_0^t)$, and therefore neglects a certain amount of statistical detail available in the voltage history it reads.

If we accept the information-theoretic approach to the submarine search problem, the conventional concept of a detection event, together with most of

the analytic modeling developed to assess the performance of sonar gear, should be suitably generalized. Consequently, much work remains to be done. To begin with, a thorough physical description of ocean phenomena relevant to acoustic propagation should be attempted. Dr. William Hurley has recently been assigned to participate in the Acoustic Detection Project and plans to investigate the feasibility of methods other than the wave equation for connecting the statistical properties of the ocean and its boundaries to the statistical properties of underwater acoustics. Next, the information process itself, given a probability distribution for the voltage history received, must be addressed in some detail to carry through the numerical comparison indicated above. Finally, we envision utilizing this framework in conjunction with some acceptable decision-making scheme to deal with certain of the Navy's major tactical necessities such as passive ranging to a target.

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